Further aspects of an interpolative quantum statistics

R. Ramanathan
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India
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Further aspects of an interpolative quantum statistics are presented. It is shown, by imposing a positivity condition on the distribution function, that this statistics subsumes Gentile's intermediate statistics and is far richer than Gentile's statistics, apart from being a genuine quantum statistics.

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I. INTRODUCTION

In a recent paper [1] I have proposed a generalized interpolative quantum statistics which interpolates nontrivially between the conventional Bose and Fermi statistical distributions through a family of distributions, one of which approximates the infinite quantum Boltzmann statistics discovered by Greenberg [2] as a consequence of the "q-mutator" algebra $a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}$ for the case of $q=0$, supplemented by the vacuum condition $a_k|0\rangle = 0$.

In an early work Gentile [3] proposed an intermediate statistics which interpolates between the Fermi and Bose distributions. This intermediate statistics is obtained by postulating a generalized Pauli principle which restricts the occupation in a quantum state by only up to k particles. Fermi statistics is recovered in this scheme as $k \to 1$ and Bose statistics is recovered as $k \to \infty$. However, the imposition of the restriction of only up to k particles in a quantum state is not invariant under a change of basis and hence cannot result in a genuine quantum statistics. A restriction which is genuinely quantum is that only up to k particles can occupy a symmetric state. Though this was known for a long time [4], no one knew how to implement this restriction in a counting scheme. Further, Gentile's intermediate statistics leads to very large violations of Fermi and Bose statistics and cannot account for very small violations which physical processes may reveal in on-going experiments [5], apart from the possible candidate process in neutral kaon decay which possibly is due to a very small violation of Bose statistics [6].

In this paper I show that the interpolative statistics [1] subsumes Gentile's statistics and is far richer in its content than its predecessor. It can thus not only accommodate very small violations of the conventional quantum statistics, but is also a genuine quantum statistics in that the counting is done in the Bose fashion without imposing any restriction on the occupation in a quantum state while the quantum state itself undergoes a deformation in phase space in a particular manner. And all this is based on very general principles of finiteness of distribution functions and their positive definiteness.

II. GENERALIZED INTERPOLATIVE QUANTUM STATISTICS

I had proposed this statistics earlier [7] by assuming the following.

(a) I assumed exotic particles other than bosons and fermions.

(b) I endowed the exotic particles with certain phase-space deformation properties as reflected in the number of phase-space cells or energy levels in any given energy interval.

(c) I adopted the Bose counting strategy to count the probability weights leading to the distribution functions.

The simplest and most economical scheme to implement the above assumptions leads to the following distribution as shown in Ref. [1]:

$$X_j^{\zeta(q)}/(X_j-1)^{[\zeta(q)-1]} = e^{\alpha+\beta E_j} . \quad (1)$$

The distribution (1) is the master distribution which is the simplest and the most general form for an interpolative statistics, provided it is finite and positive definite.

Finiteness of the distribution functions constrains the exponents in (1) to be such that $\zeta(q)$ is in the domain [0,1], otherwise we will encounter the singularity at $x_j=1$, which is not permissible for real and finite α and β, as already mentioned in Ref. [1]. Positivity of the distribution demands that the distribution be real and positive definite for all the allowed values of the exponents in (1) which constrains $X_j \geq 1$, otherwise the left-hand side of (1) can assume negative and even imaginary values for certain values of $\zeta(q)$ in the domain of its validity.

As was shown in [1], the boundary-value realizations of the master distribution (1) are the Fermi and Bose distributions, respectively, for the values $\zeta(q)=0$ and $\zeta(1)=1$.

The condition that the master distribution (1) is finite and positive definite for $\zeta(q) \in [0,1]$ and $X_j \geq 1$ implies that

$$\frac{[Z_j + \eta(N_j)]}{N_j} \geq 1 . \quad (2)$$

Since $\partial\eta(N_j)/\partial N_j = \zeta(q)$ by definition, we may choose the solution of this equation as $\eta(N_j) = \zeta(q) N_j$, setting the constant of integration to zero. The positivity condition (2) reads

$$[Z_j/N_j + \zeta(q)] \geq 1 . \quad (3)$$

Therefore, the maximum value of N_j for a given $\zeta(q)$ is from (3),

$$(N_j)_{\max} = \frac{Z_j}{[1-\zeta(q)]} . \quad (4)$$

For the Fermi case of $\zeta(q)=0$, Eq. (4) implies that $(N_j)_{\max}=Z_j$, which means that only up to one particle can occupy any quantum state or cell as (4) is true for any arbitrary N_j and Z_j, and is therefore true for $Z_j=1$, that is, for a single cell or quantum state.

For the Bose case of $\zeta(q)=1$, Eq. (4) leads to $(N_j)_{\max}=\infty$, that is, there is no restriction on the occupation of a quantum state or cell. For the intermediate case of $\zeta(q)=\frac{1}{2}$, Eq. (4) leads to $(N_j)_{\max}=2Z_j$, which implies that only up to two particles can occupy any quantum state or cell.

In general for the intermediate case of $\zeta(q)=(k-1)/k$, where k is an integer other than zero, Eq. (4) leads to $(N_j)_{\max}=kZ_j$, which implies that only up to k particles can occupy any quantum state or cell. This is precisely the generalized Pauli restriction of Gentile [3] and the present statistics subsumes Gentile's statistics in a natural fashion. But it is far richer than the Gentile statistics as we shall presently see. Moreover, this statistics in a way implements Gentile's generalized Pauli restriction for only a symmetric state, in that the counting is done as in the Bose case but with a modified number of cells. When $\zeta(q)=\frac{1}{3}$, Eq. (4) reads $(N_j)_{\max}=\frac{3}{2}Z_j$, which implies that only up to $\frac{3}{2}$ particles can occupy a quantum state or a cell. This can only have a probabilistic meaning in the sense that a particle gets shared between two adjacent states or cells. In general, when $\zeta(q)=(k-m)/k$, where $m>1$, then $(N_j)_{\max}=(k/m)Z_j$ and $m\neq k$, we will encounter a non-Gentile type of restriction on the occupancy of a quantum state. It is this feature of the statistics which makes it richer than Gentile's statistics and makes it capable of accounting for a very small violation of statistics, unlike the Gentile statistics where the deviation from conventional quantum statistics is very large.

In order to obtain the distribution function for the intermediate values of the exponent $\zeta(q)$, it should be noted that in general the master distribution presents a transcendental equation in $X_j=Z_j/N_j+\zeta(q)$, which cannot be solved analytically. However, it can be solved when $\zeta(q)$ is a simple fraction and I present a few solvable cases below.

A. Case 1

When $\zeta(q)=\frac{1}{2}$ and setting $Y_j=e^{\alpha+\beta E_j}$, Eq. (1) can be written as $X_j^2-X_j-Y_j^2=0$, whose real and positive solution leads to the distribution function as already reported in [1],

$$N_j=\frac{Z_j}{[e^{2\alpha+2\beta E_j}+\frac{1}{4}]^{1/2}}\ , \tag{5}$$

which is more Fermi-like than Bose-like but has many intermediary features.

B. Case 2

When $\zeta(q)=\frac{1}{3}$, Eq. (1) can be rewritten as $X_j^3-2X_j^2+X_j-Y_j^3=0$, whose real and positive solution leads to the following distribution function:

$$N_j=Z_j/\{[1/2Y_j^3-\tfrac{1}{27}+(Y^6_{j/4}-Y^6_{j/27}+\tfrac{10}{9}\times 81)^{1/2}]^{1/3}+[\tfrac{1}{2}Y_j^3-\tfrac{1}{27}-(Y^6_{j/4}-Y^3_{j/27}+\tfrac{10}{9}\times 81)^{1/2}]^{1/3}+\tfrac{1}{3}\}\ . \tag{6}$$

C. Case 3

When $\zeta(q)=\frac{2}{3}$, Eq. (1) can be recast as $X_j^3-X_j^2-Y_j^3=0$, whose real and positive solution leads to the following distribution function:

$$N_j=Z_j/\{[1/2Y_j^3+\tfrac{1}{27}+(Y^3_{j/27}-Y^6_{j/4}+\tfrac{10}{9}\times 81)^{1/2}]^{1/3}+[\tfrac{1}{2}Y_j^3+\tfrac{1}{27}-(Y^3_{j/27}-Y^6_{j/4}+\tfrac{10}{9}\times 81)^{1/2}]^{1/3}-\tfrac{1}{3}\}\ . \tag{7}$$

D. Case 4

When $\zeta(q)=\frac{1}{4}$, Eq. (1) can be recast as $X_j^4-3X_j^3+3X_j^2-X_j-Y_j^4=0$, whose real and positive solution leads to

$$N_j=Z_j/\{[\tfrac{1}{2}+1/2(u_1-\tfrac{3}{4})^{1/2}]+\tfrac{1}{2}[\tfrac{3}{2}-u_1+3(u_1-\tfrac{3}{4})^{1/2}+4(u^2_{1/4}+Y_j^4)^{1/2}]^{1/2}\}$$

where (8)

$$u_1=[-\tfrac{1}{2}+(\tfrac{1}{4}+Y^8_{j/27})^{1/2}]^{1/3}-[\tfrac{1}{2}+(\tfrac{1}{2}+Y^8_{j/27})^{1/2}]^{1/3}+1\ .$$

E. Case 5

Where $\zeta(q)=\frac{3}{4}$ and Eq. (1) reads $X_j^4-X_j^3-Y_j^4=0$, the real and positive solution leads to the distribution

$$N_j=Z_j/\{-[\tfrac{1}{2}+1/2(\tfrac{1}{4}+u_1)^{1/2}]+1/2[\tfrac{1}{2}-u_1+(\tfrac{1}{4}+u_1)^{1/2}+4(u^2_{1/4}+Y_j^4)^{1/2}]^{1/2}\}\ , \tag{9}$$

where

$$u_1=-\left[\left[\frac{1}{2}+\frac{\sqrt{73}}{6}\right]^{1/3}-\left[\frac{\sqrt{73}}{6}-\frac{1}{2}\right]^{1/3}\right]Y_j^{4/3}\ .$$

There may be more analytically solvable cases with larger fractional values of $\zeta(q)$ in the domain [0,1] but for the present we confine ourselves only to the above five cases.

III. CONCLUSION

Since the whole family of interpolating statistics is a genuinely quantum statistics by virtue of the indistinguishability incorporated in the counting, it is natural to seek a relationship with the deformation parameter of the Heisenberg operator algebra of Greenberg [2] in particular, and the quantum group algebra in general [9]. Though the multiparticle implications of these algebraic approaches for inducing violations of statistics is yet to be fully developed, their correspondence with the statistical distribution in the large number regime is inevitable. It is with this hindsight that I have deliberately written the deformation parameter in the present statistics as a function of the quantum group parameter q with q real and having values in the interval $-1 \leq q \leq 1$, where it has been demonstrated that the Hilbert space of vectors generated in a Fock-like representation is positive definite [2]. Though the exact relationship between the algebraic approach to the violation of statistics with the present scheme, which is based on very general assumptions, is yet to be established, for the present we may expand the parameter $\zeta(q)$ as a polynomial in q as

$$\zeta(q)=\sum_{n=0}^{\infty} C_n q^n , \tag{10}$$

which should satisfy the boundary conditions given by $\zeta(+1)=1$ and $\zeta(-1)=0$. This leads to the following conditions:

$$\sum_{n=0}^{\infty} C_n = 1 ,$$

$$\sum_{n=0}^{\infty} (-1)^n C_n = 0 , \tag{11}$$

$$C_0 \approx 0.565 .$$

For an approximate estimate of the q value from a chosen $\zeta(q)$ value we may use a quadratic approximation of (10) which is true for values $q \approx \pm 1$ and 0 as was used in Ref. [1]. A complete theory should give the exact values of the coefficients C_n in (10).

One can readily see that the whole of statistical physics can be redone for these distribution functions. The first thing that occurs to one's mind is to check on the phase-transition properties of the statistics as q varies from -1 to $+1$ or as $\zeta(q)$ varies from 0 to 1. This and many other implications of the new statistics will be reported in a subsequent paper.

It is interesting to note that in the concluding section of their paper, Jaganathan *et al.* [8] show that one of the implications of the quantum group algebra they consider is that, in its multiparticle, multimode case there seems to be an attractive collective interaction for $-1 \leq q \leq 1$ and a repulsive collective interaction for $q > 1$. Further, the population in any level is dependent on the distribution of the population in every other level. This is indeed similar to the role of the function $\eta(N_j)$ in the present statistics which can be interpreted as arising from some collective interaction due to some internal degrees of freedom [2]. However, the condition that $\eta(N_j)$ be an integer can only hold approximately and it would be a good approximation in the large number limit when the statistical distribution holds almost exactly. So, any correspondence with the present statistics with the underlying algebraic approach should be seen in this large number limit only. One hopes that this conjecture is indeed true and that there is a relationship between the algebraic approach to the new quantum statistics discovered by Greenberg and the Bose counting approach developed in this paper.

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